Problem 1.18

Calculate the curls of the vector functions in Prob. 1.15.

Solution

The three vector functions in Prob. 1.15 are

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

In Cartesian coordinates the curl of $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ is a determinant.

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\end{aligned}$$

Apply this formula to each of the vector functions.

Part (a)

$$\begin{aligned}\nabla \times \mathbf{v}_a &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(-2xz)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2)\right] \\ &= \hat{\mathbf{x}}\left[(0) - (6xz)\right] + \hat{\mathbf{y}}\left[(0) - (-2z)\right] + \hat{\mathbf{z}}\left[(3z^2) - (0)\right] \\ &= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}\end{aligned}$$

Part (b)

$$\begin{aligned}\nabla \times \mathbf{v}_b &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3zx)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right] \\ &= \hat{\mathbf{x}}\left[(0) - (2y)\right] + \hat{\mathbf{y}}\left[(0) - (3z)\right] + \hat{\mathbf{z}}\left[(0) - (x)\right] \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}\end{aligned}$$

Part (c)

$$\begin{aligned}\nabla \times \mathbf{v}_c &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy + z^2)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(2yz)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(2xy + z^2) - \frac{\partial}{\partial y}(y^2)\right] \\ &= \hat{\mathbf{x}}\left[(2z) - (2z)\right] + \hat{\mathbf{y}}\left[(0) - (0)\right] + \hat{\mathbf{z}}\left[(2y) - (2y)\right] \\ &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\end{aligned}$$